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APPROXIMATION IN PROBLEMS OF POSITION CONTROL OF PARABOLIC SYSTEMS

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The approximation of problems of position control of parabolic systems by suitable finite-dimensional control problems is discussed. The paper is closely related to the researches in [1-3].

1. We consider a system whose state at each instant t of the interval $[t_0, \vartheta]$ is characterized by a scalar function $y(t, \cdot) = y(t, x)$ defined in a domain Ω of the n-dimensional Euclidean space $\mathbf{R}^n$. The system is subject to controls u_1 and u_2 and to uncontrolled disturbances (interferences) v_1 and v_2. The system's dynamics is defined by the relations

$$\frac{\partial y(t, x)}{\partial t} = Ay(t, x) + b_1(t, x)u_1(t) - c_1(t, x)v_1(t) + \tag{1.1}$$

$$f(t, x); \quad x \in \Omega, \ t_0 < t < \vartheta$$

$$\sigma_1 \frac{\partial y(t, x)}{\partial \mathbf{v}_A} + \sigma_2(x)y(t, x) = b_2(x)u_2(t) - c_2(x)v_2(t); \ x \in \Gamma,$$

$$t_0 < t < \vartheta$$

$$y(t_0, x) = y_0(x), \quad x \in \Omega \tag{1.2}$$

$$Ay = \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + a(x)y$$

Here y_0 is a specified initial state of the system, Γ is the boundary of Ω, $\partial / \partial \mathbf{v}_A$ is the normal derivative and a_{ij}, a, b_i, c_i, f and σ_i are specified parameters; parameter σ_1 equals either zero or one $\sigma_2 = 1$ when $\sigma_1 = 0$ and $\sigma_2 \geqslant 0$ when $\sigma_1 = 1$. The quantities in (1.1) and (1.2) are assumed to satisfy certain regularity conditions (for instance, those mentioned in [2].

At each instant t the controls are constrained by $u_i(t) \in P_i(t) \subset R^{r_i}$ and the disturbances v_i have the estimates $v_i(t) \in Q_i(t) \subset R^{m_i}$, $i = 1, 2$, where $P_i(t)$ and $Q_i(t)$ are convex closed sets in the corresponding spaces, equibounded and measurable in $t \in [t_0, \vartheta]$. (Measurability and integrability are to be understood in the Lebesgue sense). For brevity we discuss only the encounter problem for system (1.1). Results similar to those given below hold for the evasion problem (see [2, 3]).

The encounter problem for system (1.1) is the following: under specified constraints on the control resources and known estimates of interference intensities find a method for forming the controls u_1 and u_2 on the feedback principle ($u_i = u_i[t, y[t, \cdot]]$, $i = 1, 2$) which would ensure that for any admissible realizations of the interference the system (1.1) is led from the initial state onto a specified state set in specified time periods in a way that the specified phase constraints are met during the control process. This problem was studied in [2, 3] wherein its mathematical formalization was presented, the necessary and sufficient solvability conditions were found and a method, similar to the extremal aiming rule [1], for constructing the resolving controls was indicated. Let us recall certain concepts from [2], needed for the subsequent presentation. We shall use the notation in [2] without explanations. Let $U(t_1, t_2, y)$ be a rule associating with each triple $\{t_1, t_2, y\}$ where $t_1 \in [t_0, \vartheta)$, $t_2 \in (t_1, \vartheta]$ and $y \in L_2(\Omega)$, a pair $\{u_1(t), u_2(t)\}$ of functions $u_1(\cdot)$, and $u_2(\cdot)$ measurable on $[t_1, t_2)$ where $u_1(t) \in P_1(t)$ and $u_2(t) \in P_2(t)$. Each such rule is called a strategy. By Δ we denote a partitioning of $[t_0, \vartheta]$ by points $t_0 = \tau_0 < \tau_1 < \ldots < \tau_{m(\Delta)} = \vartheta$, $\delta(\Delta) = \max_i (\tau_{i+1} - \tau_i)$. Let $y[t]_\Delta = y[t; t_0, y_0, U]_\Delta$, $t_0 \leqslant t \leqslant \vartheta$, be a motion of system (1.1) from position $\{t_0, y_0\}$, corresponding to strategy U and partitioning Δ (see [2]). Finally, let M and N be some sets in space $[t_0, \vartheta] \times L_2(\Omega)$. The rigorous statement of the encounter problem is as follows.

P r o b l e m 1. Construct a strategy U with the property: for any number $\varepsilon > 0$ a number $\delta > 0$ can be found for which we have

$$\rho(\{t_*, y[t_*]_\Delta\}, M) = \inf_{\{t, h\} \in M} (|t_* - t|^2 + \|y[t_*]_\Delta - h\|_\Omega^2)^{1/2} \leqslant \varepsilon$$

at some instant $t_* = t(y[\cdot]_\Delta)$ for each motion $y[t]_\Delta = y[t; t_0, y_0, U]_\Delta$, $t_0 \leqslant t \leqslant \vartheta$, with $\delta(\Delta) \leqslant \delta$ where

$$\rho(\{t, y[t]_\Delta\}, N) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_*$$

2. Let us discuss the possibility of approximating Problem 1 by suitable finite-dimensional position control problems. At first we associate a finite-dimensional controlled system of dimensionality $k \geqslant 1$ (k is an integer) with system (1.1). The state of the former system at each instant $t \in [t_0, \vartheta]$ is characterized by the vector $z^{(k)}(t) = \{z_1^{(k)}(t), \ldots, z_k^{(k)}(t)\}'$ varying by the law

$$dz^{(k)}/dt = \Lambda^{(k)} z^{(k)} + B_1^{(k)} u_1 + B_2^{(k)} u_2 - C_1^{(k)} v_1 - C_2^{(k)} v_2 + f^{(k)}(t) \tag{2.1}$$

Here u_i and v_i are controls constrained by $u_i(t) \in P_i(t)$ and $v_i(t) \in Q_i(t)$, $i = 1, 2$; the matrices in (2.1) have the form

$$\Lambda^{(k)} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_k\}$$
$$f^{(k)}(t) = \operatorname{col}\{\langle f(t, \cdot), \omega_1 \rangle_\Omega, \ldots, \langle f(t, \cdot), \omega_k \rangle_\Omega\}$$

$B_\nu^{(k)} (C_\nu^{(k)})$ is a matrix of dimension $k \times r_\nu (k \times m_\nu)$ with elements

$$b_{ij}^{(\nu)} = \langle b_{\nu j}(t, \cdot), \omega_i \rangle_{Q_\nu} \; (c_{ij}^{(\nu)} = \langle c_{\nu j}(t, \cdot), \omega_i \rangle_{Q_\nu})$$
$$\nu = 1, 2, \quad Q_1 = \Omega, \quad Q_2 = \Gamma$$

where $\{\lambda_i, \omega_i\}$ is a solution in $H^1(\Omega)$ (see [4, 5]) of the spectral problem

$$A\omega = -\lambda\omega, \; x \in \Omega; \quad \sigma_1 \frac{\partial \omega}{\partial \nu_A'} + \sigma_2 \omega = 0, \; x \in \Gamma$$

In the space $[t_0, \vartheta] \times R^k$ we associate the set

$$M^{(k)}(N^{(k)}) = \{\{t, z\} \mid t_0 \leqslant t \leqslant \vartheta,$$
$$z = \{\langle y, \omega_1\rangle_\Omega, \ldots, \langle y, \omega_k\rangle_\Omega\}', \{t, y\} \in M(N)\}$$

with set M (N). We form the vector

$$z_0^{(k)} = \{\langle y_0, \omega_1\rangle_\Omega, \ldots, \langle y_0, \omega_k\rangle_\Omega\}' = D^{(k)} y_0$$

and for system (2.1) with initial state $z^{(k)}(t_0) = z_0^{(k)}$ we consider the problem of encounter with set $M^{(k)}$ in set $N^{(k)}$(see [1]). Here it is convenient to restate this problem as follows. The strategy $U^{(k)} = U^{(k)}(t_1, t_2, z)$ is a rule associating with every triple $\{t_1, t_2, z\}$, $t_1 \in [t_0, \vartheta)$, $t_2 \in (t_1, \vartheta]$ and $z \in R^k$, a pair $\{u_1(\cdot), u_2(\cdot)\}$ of functions $u_1(t)$ and $u_2(t)$ measurable on $[t_1, t_2)$, with values in $P_1(t)$ and $P_2(t)$. The motion $z^{(k)}[t]_\Delta = z^{(k)}[t; t_0, z_0^{(k)}, U^{(k)}]_\Delta$ of system (2.1) from the position $\{t_0, z_0^{(k)}\}$, corresponding to strategy $U^{(k)}$ and partitioning Δ, is defined as the absolutely continuous solution $z^{(k)}[t]$, $t_0 \leqslant t \leqslant \vartheta$, $z^{(k)}[t_0] = z_0^{(k)}$, of Eq. (2.1) with $u_j = u_j[t]$ and $v_j = v_j[t]$, $t_0 \leqslant t \leqslant \vartheta$, where the $v_j[t]$ are some measurable functions with values in $Q_j(t)$ and on each half-open interval $[\tau_i, \tau_{i+1})$

$$\{u_1[\cdot], u_2[\cdot]\} \in U^{(k)}(\tau_i, \tau_{i+1}, z^{(k)}[\tau_i]_\Delta)$$

P r o b l e m $1^{(k)}$. Construct a strategy $U^{(k)}$ with the property: for any number $\varepsilon > 0$ a number $\delta > 0$ can be found for which the condition

$$\rho(\{t_*, z^{(k)}[t_*]_\Delta\}, M^{(k)}) = \inf_{\{t, z\} \in M^{(k)}} (|t_* - t|^2 + \| z^{(k)}[t_*]_\Delta -$$
$$z\|^2_{R^k})^{1/2} \leqslant \varepsilon$$

is fulfilled at some instant $t_* = t(z^{(k)}[\cdot]_\Delta)$ for each motion $z^{(k)}[t]_\Delta = z^{(k)}[t; t_0, z_0^{(k)}, U^{(k)}]_\Delta$, $t_0 \leqslant t \leqslant \vartheta$, with $\delta(\Delta) \leqslant \delta$, where

$$\rho(\{t, z^{(k)}[t]_\Delta\}, N^{(k)}) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_*$$

Problem $1^{(k)}$ has been studied in [1] wherein necesssary and sufficient solvability conditions were found for it and a method indicated for constructing the required controls; the case when the problem's solution can be obtained in an effective manner was distinguished. Let us point out the connection between Problem 1 and $1^{(k)}$.

T h e o r e m 2.1. Let sets M and N be closed in the metric $\|\{t, y\}\|_\alpha$ (see [2,3]). Problem 1 is solvable if and only if Problem $1^{(k)}$ is solvable for any k. Let $U^{(k)}$ be the strategy solving Problem $1^{(k)}$. We set $U_*^{(k)} = U^{(k)}(t_1, t_2, D^{(k)}y)$, $y \in L_2(\Omega)$. Then for any number $\varepsilon > 0$ a number k exists with the property: for any $k \geqslant k_0$ we can find a number $\delta = \delta(k, \varepsilon) > 0$ for which each motion $y[t]_\Delta = y\,[t;\ t_0,\ y_0,\ U_*^{(k)}]_\Delta,\ t_0 \leqslant t \leqslant \vartheta$ of system (1.1), corresponding to partitioning Δ with $\delta\,(\Delta) \leqslant \delta$, satisfies condition

$$\rho\ (\{t_*,\ y\ [t_*]_\Delta\},\ M) \leqslant \varepsilon$$

at some instant $t_* = t\ (y\ [\cdot]_\Delta)$, where

$$\rho\ (\{t,\ y\,[t]_\Delta\},\ N) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_*$$

The theorem's proof relies on the theorems on the alternative for systems (1.1) and (2.1) (see [1, 2]), on the properties of the motions of system (1.1) and on the connection between the stable sets of systems (1.1) and (2.1).

3. Let us discuss the possibility of approximating Problem 1 by suitable finite-dimensional position control problems whose construction is based on the method of finite differences. For the sake of definiteness we restrict consideration to a step domain Ω and to an implicit difference scheme. We follow the notation in [5] and assume that the grid on Ω (irregular, in general) matches the side faces of domain Ω. We write the implicit scheme for (1.1) and (1.2) (see [5]) as

$$(y_k^{\,l})_{\bar{l}} = \Lambda y_k^{\,l} + f_k^{\,l}, \quad k \in \Omega_h, \quad l = 1, \ldots, m(\Delta) \tag{3.1}$$

$$\sigma_1 \frac{\partial y_k^{\,l}}{\partial \nu} + \sigma_{2k} y_k^{\,l} = \varphi_k^{\,l}, \quad k \in \Gamma_h^{+} \tag{3.2}$$

$$y_k^{\circ} = y_{0k}, \quad k \in \Omega_h \tag{3.3}$$

Here

$$\Lambda y_k^{\,l} = \sum_{i,\,j=1}^{n} (a_{ijk}\,(y_k^{\,l})_{x_j})_{\bar{x}_i} + a_k y_k^{\,l}$$

$$y_{0k} = \frac{1}{h\,(k)} \int\limits_{\omega\,(k)} y_0\, dx$$

$$f_k^{\,l} = \frac{1}{(\tau_l - \tau_{l-1})\,h\,(k)} \int\limits_{\tau_{l-1}}^{\tau_l} \int\limits_{\omega\,(k)} (b_1 u_1 - c_1 v_1 + f)\, dx\, dt$$

If $k = (k_1, \ldots, k_p, \ldots, k_n) \in \Gamma_h^{+}$ lies on the right (left) side face Γ_p^{+} (Γ_p^{-}) orthogonal to axis Ox_p, then, in (3.2) we set $(\gamma = (k_1, \ldots k_p - 1, \ldots, k_n))$

$$\frac{\partial y_k^{\,l}}{\partial \nu} = \sum_{j=1}^{n} a_{pj\gamma} (y_\gamma^{\,l})_{x_j} \Big(- \sum_{j=1}^{n} a_{pjk} (y_k^{\,l})_{x_j} \Big)$$

$$\varphi_k^{\,l} = \frac{h_p\,(k)}{(\tau_l - \tau_{l-1})\,h\,(k)} \int\limits_{\tau_{l-1}}^{\tau_l} \int\limits_{\omega^{+}} (b_2 u_2 - c_2 v_2)\, d\Gamma\, dt$$

$$\left(\varphi_k^{\,l} = \frac{h_p\,(k)}{(\tau_l - \tau_{l-1})\,h\,(k)} \int\limits_{\tau_{l-1}}^{\tau_l} \int\limits_{\omega^{-}} (b_2 u_2 - c_2 v_2)\, d\Gamma\, dt \right)$$

if, however, it turns out that for some $k \in \Gamma_h^+$ the function φ_k^1 is specified by several values, then in (3.2) we assume that the φ_k^l for this k equals zero; $\omega^+(\omega^-)$ is the right (left) face of $\omega(\gamma)$ $(\omega(k_1,\ldots, k_p+1,\ldots, k_n))$, orthogonal to axis Ox_p.

Condition 1 (see [5-7]). The estimate

$$\max_l \sum_{\Omega_h^+} h(k)(y_k^l)^2 \leqslant C$$

holds, where C is a constant independent of the partitionings of $[t_0, \vartheta]$ and Ω, as well as of the controls u_i and v_i.

Let us rewrite scheme (3.1)-(3.2) as

$$(E-D)\ y_k^l = y_k^{l-1} + B(u,\ l) - C(v,\ l) + F \qquad (3.4)$$
$$l = 1,\ldots\ m(\Delta)$$

Here D is a linear self-adjoint nonpositive operator $H \to H$ formed with respect to the coefficients Λ, σ_1 and σ_2, and the inverse to $(E-D)$, defined on H, exists; $B(\cdot,\ l)$ $(C(\cdot,\ l))$ is a linear completely continuous operator constructed from the scheme's coefficients and defined on the set of admissible controls

$$u = \{u_1(t),\ u_2(t)\}\ (v = \{v_1(t),\ v_2(t)\}),\ \tau_{l-1} \leqslant t < \tau_l$$

with values in H; H is the space of grid functions defined on Ω_h (and equal to zero outside Ω_h), provided with the norm

$$\| y_k \|_H = \Big(\sum_{\Omega_h} h(k)(y_k)^2\Big)^{1/2}$$

With system (1.1) we associate a finite-dimensional discrete controlled system of dimensionality $|\Omega_h|$. This system's state at each instant τ_l of parttioning Δ is characterized by a vector $y_k^l \in H$ varying in accord with the recurrent law (3.4). In the space $[t_0, \vartheta] \times H$ we associate the set

$$M_h(N_h) = \Big\{\{t, y_k\} \,|\, t_0 \leqslant t \leqslant \vartheta,\ y_k = \frac{1}{h(k)} \int_{\omega(k)} y\,dx,\ k \in \Omega_h,\ \{t, y\} \in M(N)\Big\}$$

with set M (N).

The problem of encounter with M_h inside N_h for system (3.4) is formulated as follows. Strategy U_h is a rule associating with every triple $\{t_1, t_2, y_k\}$, where $t_1 \in [t_0, \vartheta)$, $t_2 \in (t_1, \vartheta]$ and $y_k \in H$, a pair $\{u_1(\cdot), u_2(\cdot)\}$ of functions $u_1(t)$ and $u_2(t)$, measurable on $[t_1,\ t_2)$, with values in $P_1(t)$ and $P_2(t)$. A collection of states $\{\overset{\circ}{y}_k,\ y_k^1, \ldots,\ y_k^{m(\Delta)}\} \subset H$ connected by the recurrence relation (3.4) is called a motion $y_h[l]_\Delta = y_h[l;\ t_0,\ y_k^\circ,\ U_h]_\Delta$ of system (3.4) from position $\{t_0, y_k^\circ\}$, corresponding to strategy U_h and partitioning Δ; at each $[\tau_i,\ \tau_{i+1})$, $u = \{u_1(\cdot),\ u_2(\cdot)\} \in U_h(\tau_i, \tau_{i+1}, y_k^i)$ and $v = \{v_1(t), v_2(t)\}$ are some measurable functions with values in $Q_1(t)$ and $Q_2(t)$, respectively.

P r o b l e m 1_h. Construct a strategy U_h with the property: for any number $\varepsilon > 0$ we can find a number $\delta = \delta(h, \varepsilon) > 0$ for which the condition

$$\rho(\{t_*, y_h^*[t_*]_\Delta\}, M_h) = \inf_{\{t, z\}\in M_h} (|t_* - t|^2 + \|y_h^*[t_*]_\Delta - z\|_H^2)^{1/2} \leqslant \varepsilon$$

is fulfilled at some instant $t_* = t(y_h[\cdot]_\Delta)$ for each motion $y_h[l]_\Delta = y[l; t_0, \overset{\circ}{y}_k, U_h]_\Delta$ corresponding to a partitioning Δ with $\delta(\Delta) \leqslant \delta$, where,

$$\rho(\{t, y_h^*[t]_\Delta\}, N_h) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_*$$

Here $y_h^*[t]_\Delta$ is an interpolation, piecewise-linear in t, of $y_h[l]_\Delta$; thus, for $t \in [\tau_l, \tau_{l+1}]$

$$y_h^*[t]_\Delta = y_h[l]_\Delta \frac{\tau_{l+1} - t}{\tau_{l+1} - \tau_l} + y_h[l+1] \frac{t - \tau_l}{\tau_{l+1} - \tau_l}$$

Without indicating the conditions under which Problem 1_h is solvable, we state the basic result right away. Let $G(v, i) = \{z \in H \mid z = y_h(\tau_i; \tau_{i+1}, y_k, u^*, v)_\Delta, u^* \in U_h(\tau_{i-1}, \tau_i, y_k), v$ ranges controls admissible on $[\tau_{i-1}, \tau_i)\}$. If $z \in H$, then by $\bar{z}$ we mean an interpolation, piecewise-constant on Ω, of z (see [5]). We note that every strategy U_h induces a strategy (rule) $U_h^*(\tau_{i-1}, \tau_i, \tau_{i+1}, y, z)$ (where τ_{i-1}, τ_i, and τ_{i+1} are elements of partitioning Δ, $y \in L_2(\Omega)$, $z \in H$) by the law

$$U_h^*(\tau_{i-1}, \tau_i, \tau_{i+1}, y, z) = \{u = \{u_1(\cdot), u_2(\cdot)\} \mid u \in U_h(\tau_i, \tau_{i+1}, z_*)$$

where z_* is ... element of $G(v, i)$, for which $\bar{z}_*$ is closest in $\|\cdot\|_\alpha$ to $y\}$. When $i = 0$ we at once set $z_* = \overset{\circ}{y}_k$.

T h e o r e m 3.1. Let Condition 1 be fulfilled and let sets M and N satisfy the hypothesis of Theorem 2.1. If we can find a sequence of concentrated grids [5] on Ω, with respect to which Problem 1_h is solvable, then Problem 1 is solvable. Let $\Delta^{(p)} \times \Omega_h^{(p)}$ be a sequence of concentrated grids on $[t_0, \vartheta] \times \Omega$, connected by the limit passage

$$\delta^\gamma \left(\max_k h^{-1/2}(k) \sum_{=1}^{n} h_i^{-1}(k)\right)^8 \leqslant \text{const}, \quad 0 < \gamma \leqslant 1$$

$$\delta(\Delta^{(p)}) \leqslant \min\{\delta(h^{(1)}, \beta_1), \ldots, \delta(h^{(p)}, \beta_p)\}, \quad \beta_p \to 0$$

and let $U_{h^{(p)}}$ be the strategy solving Problem $1_{h^{(p)}}$. Then for any number $\varepsilon > 0$ we can find a number p_0 for which each motion $y[t]_{\Delta(p)} = y[t; t_0, y_0, U_{h(p)}^*]_{\Delta(p)}$ with $p \geqslant p_0$ satisfies the condition

$$\rho_\alpha(\{t_*, y[t_*]_{\Delta(p)}\}, M) \leqslant \varepsilon$$

for some $t_* = t\ (y\,[\cdot]_{\Delta(p)})$, where

$$\rho_\alpha\ (\{t,\ y\ [t]_{\Delta(p)}\ \},\ N) \leqslant \varepsilon, \quad t_0 \leqslant t \leqslant t_*$$

The theorem's proof relies on the theorem on the alternative for system (1. 1), on the solvability theorem for Problem 1_h and on the property that the solution of difference scheme (3. 1)-(3. 3) converges to the solution of problem (1. 1) and (1. 2).

The results obtained can be used as a basis for the numerical realization of the desired control procedures on a computer.

N o t e s 1°. The results obtained hold for more general parabolic systems and also for a number of other difference schemes (see [5] , for instance).

2°. Using scheme (3. 1)-(3. 2) we can construct certain other special finite-dimensional systems (in general, no possessing the semi-group property with respect to t) for which the alternative holds and which permit a decision to be made on the choice of the first player's control in the original system (1. 1) and in the intervals between partitionings Δ; a theorem analogous to Theorem 3. 1 holds in connection with this.

3°. Strategy U_h solving Problem 1_h can be constructed as a strategy extremal to suitable sets from H.

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